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FRACTIONAL CALCULUS FRAMEWORK TO AVOID SINGULARITIES OF DIFFERENTIAL EQUATIONS

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Dedicated to Professor A.A. Kilbas on the occasion of his 60th birthday

Abstract

This paper considers the use of Riemann-Liouville fractional calculus' operators to reduce linear ordinary or partial differential equations with variable coefficients to simpler problems by means of certain commutative differential relations (known in the literature also as similarities, or transmutations). In this way, we can avoid the singularities in the original equations. In particular, here we use as an example the case of the well-known Bessel differential equation. Moreover, we obtain an extension for the known integral representation of the Bessel function.

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1. Introduction

The relevant role played by the special functions in solving numerous models in Mathematical Physics is well known, since most of these functions are solutions to linear differential equations with variable coefficients of polynomial type.

The classical methods for solving the aforementioned differential equations involve using of series with indeterminate coefficients when searching for a solution around a regular point, and the Frobenius method when obtaining solutions around regular singular points. The latter technique implies considering several particular cases, depending on the roots of the characteristic equation.

In this paper, we show the efficiency of the technique using the Riemann-Liouville operators of fractional calculus (FC) to reduce linear differential equations with variable coefficients to more elementary equations, and thus to avoid the singularities of such equations. In this way, making changes of variable implied by the fractional calculus' operators, we can represent the most frequently used special functions in terms of some basic functions, and then we can obtain solutions of the original equations in a simpler form. Such representation also allow us, in many cases, to extend the integral representation of the special functions involved.

In a more general setting, this approach is known as transmutation method for solving differential equations, or in principal, for using some integral (or integro-differential) transformation to derive the solution of a new or more complicated problem, by reducing this problem to a simpler one, whose solution is already known or exhibits simpler behaviour. For details related to notions and method of transmutations (similarities between operators) we refer the reader for example, to Delsarte and Lions [6], Hearsh [10], Dimovski [7], [8], Kiryakova [12], etc. Using operators of classical or generalized fractional calculus as transmutation techniques for solving ordinary differential equations with variable coefficients (including of higher integer or of fractional order, and generalizations of the Bessel equation) has been demonstrated in series of works, for example by Kiryakova et al. [12, Ch. 3], [15], [2], [14]; and for extending integral and differential representations of the basic classes of known special functions to wider domains or ranges of parameters, or obtaining new formulas for them, by Dimovski and Kiryakova [9], Kiryakova [12, Ch. 4], [13], etc.

In this article we consider, only as an example, the Bessel differential operator and its relationship with the Riemann-Liouville operators of FC, as well as the solution to the Bessel equation.

The paper is organized as follows: in Section 2, which is of introductory nature, we recall some definitions and properties of the Riemann-Liouville operators and give a set of properties that relate these operators with the differential operator $\delta = xD$, where D denotes the usual differential operator. The key result is shown in Section 3, and applied to the Bessel differential equation in Section 4, where we illustrate also a generalization of the known integral representation of the Bessel functions.

2. Preliminary results

In this section we remind the definitions of some operators of the fractional calculus, along with a set of their properties, used in our next discussion (For more details see, for example, Samko, Kilbas, Marichev [17]; McBride [16]; Kilbas, Srivastava, Trujillo [11]).

Definition 1. (*Riemann-Liouville (R-L) operators of fractional calculus*) Let $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, $[a, b] \subset \mathbb{R}$ and let f be a suitable real function (for example, it suffices if $f \in L_1(a, b)$). The following definitions are well known:

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \qquad (x > a) \tag{1}$$

$$(D^{\alpha}_{a+}f)(x) = D^n (I^{n-\alpha}_{a+} f)(x) \qquad (x > a) \tag{2}$$

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \qquad (x < b) \tag{3}$$

$$(D^{\alpha}_{b-}f)(x) = D^n (I^{n-\alpha}_{b-} f)(x), \qquad (x < b) \tag{4}$$

where D is the usual differential operator.

Definition 2. (*Generalized Riemann-Liouville (R-L) operators*) Under the same conditions for the function f as in the above definitions, let g be a real function such that its derivative $g'(x)$ on $[a, b]$ is greater than 0. Then:

$$(I^{\alpha}_{a+;\ g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{(g(x)-g(t))^{1-\alpha}}\, dt \qquad (x > a) \tag{5}$$

$$(D^{\alpha}_{a+;\ g}f)(x) = D^n_g (I^{n-\alpha}_{a+;\ g} f)(x) \qquad (x > a) \tag{6}$$

$$(I^{\alpha}_{b-;\ g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)}{(g(t)-g(x))^{1-\alpha}}\, dt \qquad (x < b) \tag{7}$$

$$(D^{\alpha}_{b-;\ g}f)(x) = (-1)^n D^n_g (I^{n-\alpha}_{b-;\ g} f)(x), \qquad (x < b) \tag{8}$$

where $D^n_g := \left(\frac{1}{g'(x)}\, D \right)^n$.

In particular, for the function $g(x) = x^m$, $m \in \mathbb{N}$, and $a = 0$ we obtain the following fractional integration operators (with respect to $g(x) = x^m$):

$$\begin{aligned}(I_m^\alpha f)(x) &= (I_{0+;\, x^m}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x^m - t^m)^{\alpha-1} f(t)\, dt^m \\ &= \frac{x^{m\alpha}}{\Gamma(\alpha)}\int_0^1 (1-z^m)^{\alpha-1} f(xz)\, dz^m \qquad (x>0) \qquad (9)\\ (D_m^\alpha f)(x) &= (D_{0+;\, x^m}^\alpha f)(x) = (-D_m^1)^n (I_m^{n-\alpha} f)(x) \quad (x>0). \qquad (10)\end{aligned}$$

For example, if $\alpha = 1$ and $x > 0$, we have:

$$\begin{aligned}(I_m^1 f)(x) &= \int_0^x f(t)\, dt^m = m\int_0^x f(t)\, t^{m-1}\, dt \qquad (11)\\ m(D_m^1 f)(x) &= x^{-m}(\delta f)(x), \qquad (12)\end{aligned}$$

where the known notation $\delta = xD$ is used.

Evidently, the following relations hold:

$$\begin{aligned}(D_m^1 I_m^1 f)(x) &= f(x) \qquad (13)\\ (I_m^1 D_m^1 f)(x) &= \int_0^x \frac{t^{-m}}{m}\, (\delta f)(t) dt^m = f(x) - f(0), \qquad (14)\end{aligned}$$

and the following two properties are also well known:

PROPERTY 1. *Let f be a suitable function (for instance, locally integrable or continuous) and $\alpha, \beta > 0$. Then the following relations hold (known also as semi-group properties):*

$$\begin{aligned}(I_{a+}^\alpha I_{a+}^\beta f)(x) &= (I_{a+}^{\alpha+\beta} f)(x) \qquad (15)\\ (I_m^\alpha I_m^\beta f)(x) &= (I_m^{\alpha+\beta} f)(x). \qquad (16)\end{aligned}$$

PROPERTY 2. *Let $\beta > -1$ and $\alpha > 0$ ($n-1 < \alpha < n$, $n \in \mathbb{N}$). Then*

$$D_m^\alpha x^{m\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{m(\beta-\alpha)}. \qquad (17)$$

Next, we present some properties of the FC operators that are essentially used in proving the key result in this paper.

PROPERTY 3. *Let f be a real differentiable function of order 1 in a certain interval $I \subset \mathbb{R}$, $\gamma \in \mathbb{R}$, and $m \in \mathbb{N}$. Then:*

$$(\delta\, x^\gamma) f(x) = (x^\gamma(\gamma+\delta)) f(x). \qquad (18)$$

P r o o f. $(\delta x^\gamma) f(x) = x\,(\gamma\, x^{\gamma-1} f(x) + x^\gamma D f(x)) = (x^\gamma(\delta+\gamma)) f(x)$. ∎

Property 4. *Let f be a real differentiable function of order 2 in a certain interval $I \subset \mathbb{R}$. Then:*

$$D^2 f(x) = (x^{-2}\delta(\delta - 1))f(x). \tag{19}$$

P r o o f. Using Property 3, it follows that:

$$D^2 f(x) = (x^{-1}\delta)^2 f(x) = (x^{-1}\delta x^{-1}\delta)f(x) = (x^{-2}\delta(\delta - 1))f(x). \tag{20}$$

■

Property 5. *Let $\alpha > 0$, $m \in \mathbb{N}$, and let f be a suitable function in a certain interval $I \subset \mathbb{R}$ (for instance, $f \in C^1(I)$, that is, differentiable with a continuous derivative). Then:*

$$(I_m^\alpha \delta)f(x) = ((\delta - m\alpha)I_m^\alpha)f(x) \tag{21}$$

and

$$(D_m^\alpha \delta)f(x) = ((\delta + m\alpha)D_m^\alpha)f(x), \tag{22}$$

where I_m^α and D_m^α are the generalised R-L fractional integration and differentiation operators as in (9) and (10), respectively.

P r o o f. Using Property 3,

$$(I_m^\alpha \delta)f(x) = \frac{x^{m\alpha}}{\Gamma(\alpha)} \int_0^1 (1 - z^m)^{\alpha-1} (\delta f)(xz) dz^m$$
$$= \frac{x^{m\alpha}}{\Gamma(\alpha)} \int_0^1 (1 - z^m)^{\alpha-1} \; xz \; f'(xz) dz^m$$
$$= \frac{x^{m\alpha}}{\Gamma(\alpha)} \; x \; \frac{\partial}{\partial x} \int_0^1 (1 - z^m)^{\alpha-1} f(xz) dz^m = \frac{x^{m\alpha}}{\Gamma(\alpha)} \; \delta \int_0^1 (1 - z^m)^{\alpha-1} f(xz) dz^m$$
$$= (\delta - m\alpha)\frac{x^{m\alpha}}{\Gamma(\alpha)} \int_0^1 (1 - z^m)^{\alpha-1} f(xz) dz^m = ((\delta - m\alpha)I_m^\alpha)f(x).$$

The proof of the second relationship is analogous. ■

Property 6. *Under the same hypothesis as in the above property, the following relations hold:*

$$(I_m^\alpha x^{-m}\delta)f(x) = (x^{-m}\delta I_m^\alpha)(f(x) - f(0)) \tag{23}$$

and

$$(D_m^\alpha x^{-m}\delta)f(x) = (x^{-m}\delta D_m^\alpha)(f(x) - f(0)), \tag{24}$$

where I_m^α is the Riemann-Liouville fractional integration operator in (9).

In particular, for $f(0) = 0$ we have

$$(x^{-m}\delta I_m^\alpha)f(x) = (I_m^\alpha x^{-m}\delta)f(x). \tag{25}$$

P r o o f. The proof follows from (12), (13), (14) and Property 1:

$$(x^{-m}\delta I_m^\alpha)f(x) = (mD_m^1 I_m^\alpha)[(I_m^1 D_m^1)f(x) + f(0)]$$

$$= (mD_m^1 I_m^1 I_m^\alpha D_m^1)f(x) + (mD_m^1 I_m^\alpha)f(0)$$

$$= (mI_m^\alpha D_m^1)f(x) + (mD_m^1 I_m^\alpha)f(0) = (I_m^\alpha x^{-m}\delta)f(x) + (x^{-m}\delta I_m^\alpha)f(0).$$

The proof for the second relationship is analogous. ∎

PROPERTY 7. *Let $\gamma > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $m \in \mathbb{N}$. Then, $D_m^\alpha \psi(x) = 0$ if, and only if:*

$$\psi(x) = \sum_{k=1}^{n} C_k \; x^{m(\alpha-k)}. \tag{26}$$

Practically, in this paper we use the properties of the generalized R-L operators (9), (10) when $m = 2$.

3. Commutative rule (Similarity relation)

The purpose of this section is using the fractional calculus operators to obtain a commutative rule that can serve as a tool for reducing differential equations with variable coefficients to other simpler differential equations.

LEMMA 1. *Let f be a differentiable function of order 2 in a certain interval $I \subset \mathbb{R}$, with $x \neq 0$. For the Bessel differential operator*

$$L_\alpha = D^2 + \frac{D}{x} - \frac{\alpha^2}{x^2}, \tag{27}$$

the following alternative representation holds in terms of the operator δ:

$$L_\alpha f(x) = x^{-2}(\delta-\alpha)(\delta+\alpha)\, f(x)\,. \tag{28}$$

P r o o f. Using (19), one can write

$$\begin{aligned} L_\alpha &= x^{-2}\delta(\delta-1) + \frac{x^{-1}\delta}{x} - \frac{\alpha^2}{x^2} \\ &= x^{-2}[\delta(\delta-1)+\delta-\alpha^2] = x^{-2}(\delta-\alpha)(\delta+\alpha). \end{aligned}$$

∎

Next, we present a commutative (similarity) property applicable to the Bessel differential operator L_α and a fractional differentiation operator of order $\alpha + 1/2$.

Theorem 1. *Let f be a differentiable function of order 2 in a certain interval $I \subset \mathbb{R}$, $x \neq 0$, L_α be the operator (27) and T^α - the operator defined by*

$$T^\alpha = x^\alpha D_2^{\alpha+\frac{1}{2}} \quad (\alpha > 0, \quad n-1 < \alpha + \frac{1}{2} < n, \; n \in \mathbb{N}), \tag{29}$$

where $D_2^{\alpha+\frac{1}{2}}$ is the Riemann-Liouville fractional differential operator in (10). Then the following similarity (generalized commutativity) relationship holds:

$$(T^\alpha D^2) f(x) = (L_\alpha T^\alpha)(f(x) - f(0)). \tag{30}$$

Proof. Let us consider the operator $T^\alpha = x^\alpha D_2^\beta$ $(\beta > 0; n-1 < \beta < n, n \in \mathbb{N})$, with $\beta = \beta(\alpha) \in \mathbb{R}$. Then, using properties (18),(19),(22), (24), it follows that:

$$(T^\alpha D^2) f(x) = (x^\alpha D_2^\beta x^{-2}(\delta - 1)\delta) f(x)$$

$$= (x^\alpha(\delta + 1 + 2\beta) D_2^\beta x^{-2}\delta) f(x) = (x^\alpha(\delta + 1 + 2\beta) x^{-2}\delta \; D_2^\beta)(f(x) - f(0))$$

$$= (x^{-2}(\delta + 1 - 2 - \alpha + 2\beta)(\delta - \alpha) x^\alpha \; D_2^\beta)(f(x) - f(0)).$$

Then, letting $\beta = \alpha + \frac{1}{2}$, we obtain (30). ∎

Remark 1. The so-called *hyper-Bessel differential operators* as a natural generalization of the Bessel operator (27),(28) have been introduced by Dimovski in 1966, and considered in a series of works by Dimovski and Kiryakova, as [7],[8]; [9]; [12, Ch. 3], [14]. These are singular linear differential operators of arbitrary integer order $m \geq 2$ with variable coefficients, of the form

$$B = x^{-\beta}\left[x^m \frac{d^m}{dx^m} + a_1 x^{m-1}\frac{d^{m-1}}{dx^{m-1}} + \cdots + a_{m-1} x \frac{d}{dx} + a_m\right]$$

$$= x^{-\beta} Q_m\left(x\frac{d}{dx}\right) = x^{\alpha_0}\frac{d}{dx}x^{\alpha_1}\frac{d}{dx}\ldots x^{\alpha_{m-1}}\frac{d}{dx}x^{\alpha_m},$$

with arbitrary real coefficients / parameters and m-th degree polynomial Q_m of the operator $\delta = d/dx$. For $m = \beta = 2$, $a_1 = 1, a_2 = \alpha^2$, B turns into the Bessel operator L_α, (27)-(28). *The transmutation operator T* for which the similarity (generalized commutativity) relation $TD^m f(x) = BTf(x)$ holds (under zero initial conditions), as analogue of (30), is recently called *Poisson-Dimovski transformation*, see [7], [8], [9], [12, §3.5], [14], etc. As a matter of fact, the FC operator T_α in (30) is a modification of the "Poisson transformation" arising from the representation of the Bessel function by means of the Poisson integral formula.

4. Application

In this section we apply the results from Theorem 1 to obtain explicitly the solutions to the Bessel differential equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0 \quad (x > 0). \tag{31}$$

We also show that one of the solutions extends the well-known integral representation of the Bessel function (known as Poisson integral formula) for all $\nu \in \mathbb{R}$.

First, we note that equation (31) may be written in terms of the L_α-operator (27), with $\alpha = |\nu|$, as follows:

$$(L_\alpha + 1)y(x) = 0. \tag{32}$$

Making change of variable $y(x) = T^\alpha(z(x) - z(0))$ in equation (32) yields:

$$(L_\alpha + 1)y(x) = T^\alpha[(D^2 + 1)z(x) - z(0)] = 0, \tag{33}$$

since

$$(L_\alpha + 1)y(x) = ((L_\alpha + 1)T^\alpha)(z(x) - z(0))$$
$$= (L_\alpha T^\alpha + T^\alpha)(z(x) - z(0)) = T^\alpha[(D^2 + 1)z(x) - z(0)].$$

Therefore, by using the fractional calculus operator T_α as transmutation (similarity) operator, we can avoid the singularity $x_0 = 0$, and can obtain a solution to the Bessel equation at every point $x = x_0$ (in the case $x_0 = 0$, we obtain a solution in any neighborhood of this point), by simply choosing a solution to the basic differential equation:

$$(D^2 + 1)z(x) = z(0), \tag{34}$$

that is,

$$z(x) = C_1 \sin(x) + z(0) \quad (C_1 \text{ a real constant}). \tag{35}$$

We shall choose the solution $z_1(x) = \sin(x)$, which yields the following solutions for the Bessel equation (31), valid for any real value of ν:

$$\begin{aligned} y_1(x) &= (x^{|\nu|} D_2^{|\nu|+\frac{1}{2}}) \sin(x) \\ &= \frac{2^{1-n}\, x^{|\nu|}}{\Gamma(n - (|\nu| + \frac{1}{2}))} \left(x^{-1}\, D\right)^n \int_0^x (x^2 - t^2)^{n-|\nu|-\frac{3}{2}}\, t\, \sin t\, dt, \end{aligned} \tag{36}$$

with $x > 0$ and $n - 1 < |\nu| + \frac{1}{2} < n$.

Up to a constant multiplier, the expression (36) represents the Bessel function of order ν, and, given the appropriate restrictions, matches its integral representation. For example, for $\alpha = |\nu| < \frac{1}{2}$ it holds that:

$$y_1(x) = (x^\alpha D_2^{\alpha+\frac{1}{2}} I_2^1 D_2^1)(\sin(x)) = \left(\frac{x^\alpha}{2} I_2^{\frac{1}{2}-\alpha} x^{-1}\right)(\cos(x)), \quad (37)$$

as a variant of the Poisson integral formula. See, for instance, Abramowitz and Stegun [1].

A generalization of (37) for the so-called hyper-Bessel functions as solutions of hyper-Bessel differential equations $(B+1)y(x) = 0$ of the form analogous to (31)-(32), called generalized Poisson integral formula, can be found in [9], [12, §3.7], [13].

Finally, let us point out that various procedures exist for obtaining a linearly independent second solution $y_2(x)$ from $y_1(x)$ for the Bessel equation (31). The most natural method is to reduce the Bessel equation to a first order linear equation, for which we already know a solution, and then to solve that equation directly. We can also extend the method used for obtaining $y_1(x)$:

$$\begin{aligned} y_2(x) &= y_1(x) \int \frac{1}{x\,(y_1(x))^2}\, dx \\ &= \left[\left(x^\alpha D_2^{\alpha+\frac{1}{2}}\right)\sin x\right] \int \frac{1}{x^{2\alpha+1}\left(D_2^{\alpha+\frac{1}{2}}\sin x\right)^2}\, dx. \end{aligned} \quad (38)$$

Additionally, keeping in mind property (26), it directly follows that every solution to

$$(D^2+1)z(x) = z(0) + x^{(2\alpha-1)}, \quad (39)$$

is also a solution to the Bessel equation (32), except for $x = 0$, as long as $z(x)$ displays suitable behavior at $x = 0$.

Finding a particular solution to equation (39) is easy under the corresponding restrictions:

$$z_p(x) - z_p(0) = \int_0^x t^{(2\alpha-1)}\, (\sin(x)\cos(t) - \cos(x)\sin(t))\, dt, \quad (40)$$

therefore the second linearly independent solution to the Bessel equation is:

$$y_2(x) = \left(x^\alpha D_2^{\alpha+\frac{1}{2}}\right) \int_0^x t^{(2\alpha-1)}\, (\sin(x)\cos(t) - \cos(x)\sin(t))\, dt. \quad (41)$$

Remark 2. Let us point out that many other authors have studied generalizations of the classical special functions in the framework of the fractional calculus, and they have applied techniques of transmutation, similar to the used in this paper to solve the differential equation for the hypergeometric function ${}_0F_1$, to find solutions of certain fractional differential equations involving generalized Bessel differential operators, etc. See, for instance, [9], [12], [13], [2], and [3], [4], [5].

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